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LIKELIHOOD RATIO TESTS FOR RESTRICTED FAMILIES

by

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ABSTRACT

Likelihood ratio tests for the problem: $H_0: F \in \mathcal{F}_1$ versus $H_1: F \in \mathcal{F} - \mathcal{F}_1$ are defined for certain nonparametric families of distributions $\mathcal{F}$ and $\mathcal{F}_1$. In particular the likelihood ratio test is defined and shown to be unbiased when $\mathcal{F}_1$ denotes the exponential distributions (possibly truncated) and $\mathcal{F}$ denotes the distributions with increasing failure rate. Comparisons are made with competing tests. The problem of testing for increasing failure rate average is also examined.

LIKELIHOOD RATIO TESTS FOR RESTRICTED FAMILIES

1. Introduction

Tests for composite hypotheses having optimal properties for finite samples
have been obtained for various special problems by an important principle due to
Neyman and Pearson (1928, 1933) called the likelihood ratio principle. This principle
leads to the likelihood ratio test. Asymptotic properties of this test for
parametric families of distributions can be found in Wilks (1962). Recently
a conditional likelihood ratio test has been proposed for testing for trend in a
stochastic process of Poisson type [Boswell (1966)]. This is a departure from
the standard literature in that the underlying family of distributions considered
is essentially nonparametric. The main result obtained is the asymptotic distribution
of the likelihood ratio under the null hypothesis of no trend.

We consider likelihood ratio tests for certain geometrically restricted
families of distributions. For example, let

$$\mathcal{F}_0 = \left\{ F \mid F(0) = 0 \quad \text{and} \quad \frac{-\log\,[1-F(x)]}{x} \quad \text{nondecreasing in} \quad x \geq 0 \right\}.$$

Then $\mathcal{F}_0$ is known as the IFRA (for increasing failure rate average) family of
distributions. These distributions play an important role in the mathematical
theory of reliability [Birnbaum, Esary, and Marshall (1966)]. However, not only
is the family nonparametric but there is no sigma-finite measure relative to which
all $F \in \mathcal{F}_0$ are absolutely continuous. Hence, the usual concept of maximum
likelihood estimate does not suffice. Kiefer and Wolfowitz (1956, p. 893) propose
a generalization of the maximum likelihood estimate concept which we adopt. Let
F_1, F_2 $\in \mathcal{F}$ and let $f(\cdot;\, F_1,\, F_2)$ denote

the Radon-Nikodym derivative of F_1 with respect to the measure induced by $F_1 + F_2$.

Definition 1

$\hat{F}$ is called the <u>maximum likelihood estimate</u> relative to $\mathcal{F}$ if $\hat{F}$ satisfies

$$\sup_{F \in \mathcal{F}} \prod_{i=1}^{n} \left[\frac{f(X_i;\, F,\, \hat{F})}{1 - f(X_i;\, F,\, \hat{F})} \right] = 1 \, ,$$

where $\underline{X} = (X_1, X_2, \ldots, X_n)$ is a random sample from some $F \in \mathcal{F}$.

This definition is easily seen to coincide with the usual definition when the family $\mathcal{F}$ is dominated by a sigma-finite measure.

Now consider the problem of testing $H_0: F \in \mathcal{F}_0$ against the alternative $H_1: F \in \mathcal{F} - \mathcal{F}_0$ where $\mathcal{F}_0 \subset \mathcal{F}$. Let $\hat{F}_0 \, (\hat{F})$ denote the maximum likelihood estimate relative to $\mathcal{F}_0 \, (\mathcal{F})$ in the sense of definition 1. We define the likelihood ratio statistic $\Lambda_n(\underline{X})$ based on a random sample $\underline{X}$ as follows:

Definition 2

$\Lambda_n(\underline{X})$ is called the <u>likelihood ratio statistic</u> where

$$\Lambda_n(\underline{X}) = \prod_{i=1}^{n} \left[\frac{f(X_i;\, \hat{F}_0,\, \hat{F})}{1 - f(X_i;\, \hat{F}_0,\, \hat{F})} \right] \, .$$

We will be concerned with the properties of $\Lambda_n(\underline{X})$ for various restricted families of distributions $\mathcal{F}_0 \subset \mathcal{F}$.

2. IFRA Distributions

Let $\mathcal{F}_0 = \{ F \mid F(0) = 0$ and $\frac{-\log\,[1 - F(x)]}{x} \uparrow$ in $x \geq 0 \}$ and $\underline{X} = (X_1, X_2, \ldots, X_n)$ denotes a random sample from F. We claim that the maximum likelihood

estimate (MLE) $\hat{F}_0$ relative to $\mathcal{F}_0$ puts mass at each of the sample observations. To see this suppose $\mathcal{F}^* = \{F_1, F_2\}$ where the F_1 probability of the observation X_i is $F_1\{X_i\} > 0$ for $i = 1, 2, \ldots, n$ and $F_2\{X_k\} = 0$ for some k ($1 \leq k \leq n$) . From definition 1 it follows that F_1 is MLE in $\mathcal{F}^*$. Since $F \epsilon \mathcal{F}_0$ can put mass at a countable number of points we may restrict attention to those $F \epsilon \mathcal{F}_0$ putting mass at sample points; i.e., F absolutely continuous with respect to $\lambda + \mu$ where λ is Lebesque measure and $\mu\{A\}$ equals the number of sample points in A . The likelihood becomes

$$L_n(\underline{X} \mid F) = \prod_{i=1}^{n} F \{X_i\} .$$

Proschan and Marshall (1967) have obtained the MLE under the IFRA assumption. From the definition of IFRA distributions we see that

$$(2.1) \qquad L_n(\underline{X} \mid F) = \prod_{i=1}^{n} \left[\exp(- \lambda_{i-1} X_i) - \exp(- \lambda_i X_i) \right]$$

where $0 \leq \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_n$. We maximize likelihood subject to these restrictions by letting $\lambda_0 = 0$ and $\lambda_n = +\infty$. Letting $\Delta \lambda_j = \lambda_j - \lambda_{j-1}$ where $\lambda_0 = 0$ and $\lambda_n = +\infty$, we see that (2.1) becomes

$$(2.2) \qquad L_n(\underline{X} \mid F) = \prod_{i=1}^{n-1} \left[\exp(- \Delta \lambda_i \sum_{i+1}^{n} X_j) \left[1 - \exp(- \Delta \lambda_i X_i) \right] \right] .$$

Maximizing (2.2) subject to $\Delta \lambda_i \geq 0$ ($1 \leq i \leq n$) we see that

$$\frac{X_i}{\sum\limits_{j=i}^{n} X_j} \geq \frac{\psi(X_i)}{\sum\limits_{j=i}^{n} \psi(X_j)} = \frac{Y_i^*}{\sum\limits_{j=i}^{n} Y_j^*} \quad .$$

Since $h(x) = x(1-x)^{\frac{1}{x} - 1}$ is increasing in x $(0 \leq x \leq 1)$, it follows that

$$\prod_{i=1}^{n-1} h\left(\frac{X_i}{\sum\limits_{i}^{n} X_j}\right) \underset{st}{\geq} \prod_{i=1}^{n-1} h\left(\frac{Y_i}{\sum\limits_{i}^{n} Y_j}\right)$$

where $(Y_1 \leq Y_2 \leq \dots \leq Y_n)$ is an independent ordered sample from G. (2.6) follows immediately and the proof of (2.7) is similar. $\|$

We say that $F_1 \underset{*}{\leq} F_2$ (i.e., F_1 is starshaped with respect to F_2) if $\frac{F_2^{-1} F_1(x)}{x}$ is nondecreasing for $x \geq 0$. From the proof of theorem 2.1 it follows that $F_1 \underset{*}{\leq} F_2$ implies

$$P_{F_1}\{\Lambda_n(\underline{X}) \leq c_\alpha\} \leq P_{F_2}\{\Lambda_n(\underline{X}) \leq c_\alpha\} \quad .$$

Hence the power of the likelihood ratio test is greater at F_2 than at F_1 when $F_1 \underset{*}{\leq} F_2$. Percentage points for $-\log \Lambda_n(\underline{X})$ are given in Table 1.

Of course there are many unbiased tests of the IFRA hypothesis. Marshall, Walkup and Wets (1966) have characterized the class of all such tests. These are just the tests based on functions $f(x_1, x_2, \dots, x_n)$ having the properties:

1) f is homogenous;

2) $\sum\limits_{i=1}^{j} x_i \frac{\partial f(x_1, x_2, \dots, x_n)}{\partial x_i} \geq 0$ for $j = 1, 2, \dots, n-1$ and all $x_1 \geq x_2 \geq \dots \geq x_n \geq 0$.

The test associated with f would reject H_o if

$$f(X_1, X_2, \ldots, X_n) \geq c$$

where c is some suitable critical number and $X_1 \geq X_2 \geq \cdots \geq X_n$ are the usual order statistics labelled in reverse order.

3. IFR Distributions

Let $\mathscr{F} = \{F \mid F(0) = 0$ and $- \log [1 - F(x)]$ is convex for $x \geq 0\}$. This is the class of IFR (for increasing failure rate) distributions. Proschan and Pyke (1965) have proposed a test for constant versus increasing failure rate. Their test is based on a statistic considered by M. G. Kendall (1938) and H. B. Mann (1945) and is essentially a rank test for trend. Proschan and Pyke show that their test is unbiased, has good large sample properties, and is competitive in this sense with certain parametric tests when the unknown distribution lies within some specified parametric family. However, sampling experiments indicate that their test does not have good discriminating power for relatively small samples. This is perhaps to be expected since their test does not use all of the information in the sample.

M. Boswell (1966) studied a similar problem concerning Poisson type processes. His statistic based on a conditional maximum likelihood ratio test is essentially the same as the likelihood ratio statistic studied in this section. The main result in Boswell's paper is a derivation of the asymptotic distribution of his test statistic. In contrast, we concentrate on small sample results.

Since IFR distributions can have a jump at the right hand end of their interval of support it is clear from definition 1 that we need only consider

estimators absolutely continuous with respect to Lebesgue measure on $[0, X_n)$ with a jump at X_n (see Barlow and Proschan, (1965), p. 26). Hence

$$L_n(\underline{X} \mid F) = \left[\prod_{i=1}^{n-1} f(X_i) \right] \; F\{X_n\}$$

where f is the density of F on $[0, X_n)$. Since

$$1 - F(x) = \exp\left[- \int_0^x r(u)du \right] \quad \text{where} \quad r(u) = f(u)/[1 - F(u)]$$

for $0 \le u < X_n$, we may write

$$f(x) = r(x) \; e^{- \int_0^x r(u)du} \qquad\qquad 0 \le x < X_n$$

and

$$F\{X_n\} = e^{- \int_0^{X_n} r(u)du} \qquad .$$

Hence

$$(3.1) \qquad \log L_n(\underline{X} \mid F) = \sum_{i=1}^{n-1} \log r(X_i) - \sum_{i=1}^{n} \int_0^{X_i} r(u)du \quad .$$

The problem of maximizing (3.1) subject to $r(x)$ nondecreasing was solved by Grenander (1956) and independently by Marshall and Proschan (1965). They show that the problem can be reduced to maximizing

$$\sum_{i=1}^{n-1} \log r(X_i) - \sum_{i=1}^{n-1} (n - i) \; (X_{i+1} - X_i) \; r(X_i)$$

subject to $r(X_1) \le r(X_2) \le \ldots \le r(X_{n-1})$. The maximum likelihood estimates are

$$\hat{r}_n(X_i) = \min_{v \ge i+1} \max_{u \le i} \left[\frac{(v-u)}{(n-u)(X_{u+1} - X_u) + \cdots\cdots + (n-v+1)(X_v - X_{v-1})} \right]$$

for $i = 1, 2, \ldots, n-1$. The maximum likelihood is

$$(3.2) \qquad L_n(\underline{X} \mid \hat{F}) = \left[\prod_{i=1}^{n-1} \hat{r}(X_i) \right] e^{-(n-1)} .$$

The exponent on e can be easily verified using the definition of $\hat{r}$ and observing that

$$\sum_{i=1}^{n} \int_0^{X_i} \hat{r}(u)du = \sum_{i=1}^{n-1} (n-i) \int_{X_i}^{X_{i+1}} \hat{r}(u)du .$$

Let $\mathscr{F}_0 = \{F \mid F(0) = 0, F(x) = 1 - e^{-\lambda x} \text{ for } x < T \text{ and } F(T) = 1, \lambda > 0, T > 0\}$

Then $\mathscr{F}_0$ denotes the class of exponential distributions with possible truncation on the right. Consider now the problem of testing H_0: $F \in \mathscr{F}_0$ versus H_1: $F \in \mathscr{F} - \mathscr{F}_0$. The choice of H_0 was determined by the fact that the MLE's $\hat{F}_0$ and $\hat{F}$ are both absolutely continuous with respect to Lebesgue measure in $[0, X_n)$ and place mass at X_n. The likelihood under H_0 will be

$$L_n(\underline{X} \mid F_0) = \left[\prod_{i=1}^{n-1} \lambda e^{-\lambda X_i} \right] e^{-\lambda X_n}$$

and the maximum likelihood will be

$$(3.3) \qquad L_n(\underline{X} \mid \hat{F}_0) = \left(\frac{n-1}{\sum_i^n X_i} \right)^{n-1} e^{-n} .$$

According to definition 2, the likelihood ratio statistic for testing for truncated exponentiality versus IFR and not truncated exponentiality will be

(3.4) $\qquad \Lambda_n^* (\underline{X}) = \dfrac{(n-1)^{n-1}}{\left(\sum\limits_1^n X_i\right)^{n-1} \prod\limits_{i=1}^{n-1} \hat{r}(X_i)}$.

If $\dfrac{1}{(n-1)(X_2 - X_1)} \leq \dfrac{1}{(n-2)(X_3 - X_2)} \leq \cdots \leq \dfrac{1}{(X_n - X_{n-1})}$ so that

$$\hat{r}(X_i) = \frac{1}{(n-i)(X_{i+1} - X_i)} \qquad i = 1, 2, \ldots, n-1 ,$$

then (3.4) becomes

$$\Lambda_n^{**}(\underline{X}) = \left(\frac{n-1}{\sum\limits_1^n X_i}\right)^{n-1} \prod_{i=1}^{n-1} (n-i)(X_{i+1} - X_i) .$$

As in section 2 we consider the test, φ^* , which rejects H_o when

$$\Lambda_n^* (\underline{X}) \leq c_\alpha$$

where c_α is determined by

$$P_G \{\Lambda_n^* (\underline{Y}) \leq c_\alpha\} = \alpha .$$

The asymptotic distribution of $\Lambda_n^* (\underline{Y})$ can be found in Boswell (1966, p. 1572). A table of percentage points obtained using Monte Carlo methods is contained in Table 2.

4. Unbiasedness of the Likelihood Ratio Test for IFR

Like the Proschan-Pyke test, the likelihood ratio test has greater power under the alternative than under the null hypothesis. To show this we need to introduce some auxilliary results.

Given a sequence of nonnegative real numbers $\{z_i\}_{i=1}^{n}$, plot $\sum_1^i z_j$ versus i and interpolate linearly between $(0, 0)$, $(1, z_1)$, ... $(n, z_1 + ... + z_n)$. Let $\bar{z}_1 \geq \bar{z}_2 \geq ... \geq \bar{z}_n$ denote the slopes of the least concave majorant to this graph in successive intervals. This operation converts the original sequence into a nonincreasing sequence and will be useful later on. For convenience, call $\bar{z}_1 \geq \bar{z}_2 \geq ... \geq \bar{z}_n$ the Brunkized sequence after D. Brunk (see D. Brunk et. al. (1955)). Note that $\bar{z}_1 \geq \bar{z}_2 \geq ... \geq \bar{z}_n$ can also be obtained by successive averaging of the original sequence until it becomes nonincreasing.

We say that $H(z_1, z_2, ..., z_n)$ is a <u>Schur function</u> if

$$(z_i - z_j) \left(\frac{\partial H}{\partial z_i} - \frac{\partial H}{\partial z_j} \right) \geq 0$$

for all vectors $\underline{z}$ and $i, j = 1, 2, ..., n$. This concept is needed in the following useful lemma.

<u>Lemma 1</u>

Let $(z_1, z_2, ..., z_n)$ and $(z_1', z_2', ..., z_n')$ denote two nonnegative sequences such that

$$\sum_1^r z_i \geq \sum_1^r z_i' \qquad \text{for } r = 1, 2, ..., n - 1$$

and

$$\sum_1^n z_i = \sum_1^n z_i' \quad .$$

Then the inequalities are preserved under Brunkization;

i.e.,
$$\sum_{i=1}^{r} \overline{z}_i \geq \sum_{i=1}^{r} \overline{z_i'} \qquad (r = 1, 2, \ldots, n - 1)$$

(i)
$$\sum_{i=1}^{n} \overline{z_i} = \sum_{i=1}^{n} \overline{z_i'} \ .$$

If H is a Schur function then

$$\text{(ii)} \qquad H(\overline{z}_1, \overline{z}_2, \ldots, \overline{z}_n) \geq H(\overline{z_1'}, \overline{z_2'}, \ldots, \overline{z_3'}) \ .$$

Proof:

(i) is obvious since the least concave majorant to the $\{z_i\}$ sequence lies above the least concave majorant to the $\{z_i'\}$ sequence.

Since (i) holds, $\overline{z}_1 \geq \overline{z}_2 \geq \cdots \geq \overline{z}_n$; $\overline{z_1'} \geq \cdots \geq \overline{z_n'}$ and $\sum_1^n \overline{z}_i = \sum_1^n \overline{z_i'}$

we have (ii) by the Schur, Ostrowski theorem (see Ostrowski (1952)). //

Theorem 4.1

If $G^{-1}F(x)$ is convex for $x \geq 0$, $G(0) = F(0) = 0$ and $\underline{X}$ ($\underline{Y}$) denotes a random sample from $F(G)$, then

$$\Lambda_n^* (\underline{X}) \underset{st}{\leq} \Lambda_n^* (\underline{Y}) \ .$$

Remark

This proves the likelihood ratio test Λ_n^* is unbiased since if F is IFR and $G(x) = 1 - e^{-x}$ for $x \geq 0$, then $G^{-1}F$ is convex on $x \geq 0$.

Proof:

Let $Y_i^* = G^{-1}F(X_i)$ and note $Y_i^* \underset{st}{=} Y_i$.

Let
$$z_i = \frac{(n-i)(X_{i+1} - X_i)}{\displaystyle\sum_1^{n-1} (n-i)(X_{i+1} - X_i)} \quad \text{and}$$

$$z_i' = \frac{(n-i)(Y_{i+1}^* - Y_i^*)}{\displaystyle\sum_1^{n-1} (n-i)(Y_{i+1}^* - Y_i^*)} \quad .$$

Since $Y_i^* = G^{-1}F(X_i)$ and $G^{-1}F$ is convex

$$\frac{(n-i)(Y_{i+1}^* - Y_i^*)}{(n-i)(X_{i+1} - X_i)}$$

is increasing in $i = 1, 2, \ldots, n$. It follows from lemma 3.7 (i) of Barlow and Proschan (1966) that

$$\frac{\displaystyle\sum_1^r z_i'}{\displaystyle\sum_1^r z_i} \geq \frac{\displaystyle\sum_1^{n-1} z_i'}{\displaystyle\sum_1^{n-1} z_i} = 1$$

and hence $\displaystyle\sum_1^r z_i \geq \sum_1^r z_i'$ for $r = 1, 2, \ldots, n-1$. Let $\{\bar{z}_i\}$ and $\{\overline{z_i'}\}$ denote the Brunkized estimates of $\{z_i\}$ and $\{z_i'\}$ respectively. Let

$$-H(x_1, x_2, \ldots, x_{n-1}) = \prod_{i=1}^{n-1} x_i$$

and note that H is a Schur function. Since $\{z_i\}$ and $\{z_i'\}$ satisfy the hypotheses of lemma 1, it follows that

$$-H(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_{n-1}) \le -H(\overline{z_1}, \overline{z_2}, \ldots, \overline{z_{n-1}}) \ .$$

Hence

$$(4.1) \quad \frac{1}{\left[\sum\limits_{1}^{n-1} (n-i)(X_{i+1} - X_i)\right]^{n-1} \prod\limits_{1}^{n-1} \hat{r}(X_i)} \le \frac{1}{\left[\sum\limits_{1}^{n-1} (n-i)(Y_{i+1}^* - Y_i^*)\right]^{n-1} \prod\limits_{1}^{n-1} \hat{r}(Y_i^*)} \ .$$

Since

$$\frac{\sum\limits_{0}^{r} (n-i)(Y_{i+1}^* - Y_i^*)}{\sum\limits_{0}^{r} (n-i)(X_{i+1} - X_i)} \le \frac{\sum\limits_{1}^{n} Y_i^*}{\sum\limits_{1}^{n} X_i}$$

for $(1 \le r \le n - 1)$ by lemma 3.7 (i') of Barlow and Proschan (1966) it follows that

$$(4.2) \quad \frac{\sum\limits_{1}^{n-1} (n-i)(X_{i+1} - X_i)}{\sum\limits_{1}^{n} X_i} \le \frac{\sum\limits_{1}^{n-1} (n-i)(Y_{i+1}^* - Y_i^*)}{\sum\limits_{1}^{n} Y_i^*} \ .$$

(4.1) and (4.2) together imply

$$\Lambda_n^*(\underline{X}) \le \Lambda_n^*(\underline{Y}^*) \ .$$

The theorem follows from $(Y_1, Y_2, \ldots, Y_n) \underset{st}{=} (Y_1^*, Y_2^*, \ldots, Y_n^*)$. //

Marshall, Walkup and Wets (1966) have characterized the class of unbiased tests for constant failure rate versus nondecreasing failure rate. These are based on functions $h(x_1, x_2, \ldots, x_r)$ satisfying the conditions

i) h is homogeneous;

ii) $\sum_{i=1}^{j} (x_i - x_{j+1}) \dfrac{\partial h(x_1, \ldots, x_n)}{\partial x_i} \geq 0 \quad j = 1, 2, \ldots, n-1$

for all $x_1 > x_2 > \ldots > x_n > 0$. The corresponding test consists of rejecting exponentiality if $h(X_1, X_2, \ldots, X_n) \leq c$ where c is a suitable critical number and $X_1 \geq X_2 \geq \ldots \geq X_n$ are the order statistics labelled in reverse order.

5. Distribution of the Maximum Likelihood Ratio Statistic Under the Exponential Assumption

From the computations in Boswell (1966) it is clear that the distribution of Λ_n^* , even under the null hypothesis, is exceedingly complicated. For this reason we have had to use Monte Carlo methods to obtain the percentage points tabulated in Table 2. However, the distribution of Λ_n^* under H_o is quite smooth as we show in

Theorem 5.1

The likelihood ratio statistic Λ_n^* has a nonincreasing density on $(0, 1)$ under the exponential assumption.

Proof:

Let $0 = W_0 \leq W_1 \leq \ldots \leq W_n$ denote an ordered sample from the uniform distribution on $(0, 1)$. Let

$$U_i = W_i - W_{i-1} \qquad i = 1, 2, \ldots, n-1 .$$

Then the random vector $(U_1, U_2, \ldots, U_{n-1})$ has joint density

$$
h(u_1, u_2, \ldots, u_{n-1}) = \begin{cases} (n-1)! & \text{for } u_i \geq 0 \\ & i = 1, 2, \ldots, n-1 \\ & 0 \leq u_1 + u_2 + \ldots + u_{n-1} \leq 1 \\ 0 & \text{otherwise.} \end{cases}
$$

Let $(\bar{U}_1, \bar{U}_2, \ldots, \bar{U}_{n-1})$ denote the modified vector $(U_1, U_2, \ldots, U_{n-1})$ after Brunkization and subject to $\bar{U}_1 \geq \bar{U}_2 \geq \cdots \geq \bar{U}_{n-1}$.

The likelihood ratio statistic

$$
\Lambda_n^*(\underline{Y}) = \frac{1}{\left(\dfrac{\sum\limits_1^n Y_i}{n-1}\right)^{n-1} \prod\limits_1^{n-1} \hat{r}(Y_i)}
$$

is distributed as $(n-1)^{n-1} \prod\limits_1^{n-1} \bar{U}_i$ under the exponential assumption. Notationally it will be convenient to replace $n-1$ by n. Hence we need only prove that

$$
P\left\{ \prod_{i=1}^n \bar{U}_i \leq z \right\}
$$

is concave in $z \in (0, 1)$. Let I denote the usual indicator set function and observe that

$$
P\left\{ \prod_{i=1}^n \bar{U}_i \leq z \right\} = n! \iint\limits_{\substack{u_i \geq 0 \\ 0 \leq u_1 + \ldots + u_n \leq 1}} \int I\left[\bar{u}_1 \ldots \bar{u}_n \leq z\right] du_1 \ldots du_n \; .
$$

Integrating out on u_n we have

$$P\left\{\prod_1^n \bar{U}_i \leq z\right\} = n! \underset{\substack{u_i \geq 0 \\ u_1 + \ldots + u_{n-1} \leq 1}}{\iint \cdots \int} f(u_1, \ldots, u_{n-2}; z) du_1 \ldots du_{n-1}$$

where

$$f(u_1, \ldots, u_{n-1}; z) = \int_{0 \leq u_n \leq 1 - u_1 - \ldots - u_{n-1}} I\left[\bar{u}_1 \ldots \bar{u}_n \leq z\right] du_n$$

$$= \min\left[1 - u_1 - \ldots - u_{n-1}, u_n(z)\right]$$

and $u_n(z)$ is the solution of $z = \bar{u}_1 \ldots \bar{u}_{n-1} \bar{u}_n$

for fixed $u_1, u_2, \ldots, u_{n-1}$.

We claim that z is a strictly increasing convex function of u_n and, therefore, that u_n is a strictly increasing <u>concave</u> function of z. It follows that $f(u_1, \ldots, u_{n-1}; z)$ is a concave function of z for fixed $(u_1, u_2, \ldots, u_{n-1})$. Hence

$$P\left\{\prod_1^n \bar{U}_i \leq z\right\} = n! \underset{\substack{u_i \geq 0 \\ u_1 + \ldots + u_{n-1} \leq 1}}{\iint \cdots \int} f(u_1, \ldots, u_{n-1}; z) du_1 \ldots du_{n-1}$$

is a concave function of z.

To show $z = \bar{u}_1 \ldots \bar{u}_{n-1} \bar{u}_n$ is a convex function of u_n, define $(\bar{u}_1^*, \ldots, \bar{u}_{n-1}^*)$ to be the Brunk modification of $(u_1, u_2, \ldots, u_{n-1})$ subject to $\bar{u}_1^* \geq \bar{u}_2^* \geq \ldots \geq \bar{u}_{n-1}^*$. Clearly z is piecewise convex for u_n in the intervals $[0, \bar{u}_{n-2}^*], [\bar{u}_{n-2}^*, \bar{u}_{n-3}^*], \ldots, [1 - \bar{u}_1^* - \ldots - \bar{u}_{n-1}^*, \bar{u}_1^*]$. It is therefore sufficient to show that z has a continuous derivative in u_n. We show that the right and left hand derivatives at $u_n = \bar{u}_{n-1}^*$ are equal. For $u_n < \bar{u}_{n-1}^*$

$$\frac{dz}{du_n} = \bar{u}_1^* \ldots \bar{u}_j^* \left(\frac{u_{j+1}+\ldots+u_{n-1}}{n-1-j} \right)^{n-j-1} .$$

For $u_{n-1}^* < u_n < u_{n-2}^*$

$$\frac{dz}{du_n} = \bar{u}_1^* \ldots \bar{u}_j^* \left(\frac{u_{j+1}+\ldots+u_n}{n-j} \right)^{n-j-1} .$$

For $\bar{u}_n = \bar{u}_{n-1}^*$, obviously

$$\left(\frac{u_{j+1}+\ldots+u_{n-1}}{n-1-j} \right) = \left(\frac{u_{j+1}+\ldots+u_{n-1}+u_n}{n-j} \right) \qquad . \; ||$$

For $n = 2$ and $n = 3$ it is a straightforward computation to obtain the distribution of $\Lambda_n^*(\underline{Y})$. Clearly, for $n = 2$

$$\Lambda_2^*(\underline{Y}) = U_1$$

and the likelihood ratio is uniformly distributed on $(0, 1)$.

For $n = 3$

$$\Lambda_3^*(\underline{Y}) = 4 \, \bar{U}_1 \, \bar{U}_2 = \begin{cases} 4 \, U_1 \, U_2 & \text{if } U_1 \geq U_2 \\ (U_1 + U_2)^2 & \text{if } U_1 \leq U_2 . \end{cases}$$

Hence

$$P_G \left\{ \Lambda_3^*(\underline{Y}) \leq u \right\} = \iint_{\substack{\{u_1 u_2 \leq \frac{u}{4}\} \\ u_1 \geq u_2}} 2 \; du_1 \; du_2 + \iint_{\substack{\{u_1+u_2 \leq \sqrt{u}\} \\ u_1 \leq u_2\}}} 2 \; du_1 \; du_2$$

and

$$P_G \left\{ \Lambda_3^*(\underline{Y}) \leq u \right\} = \frac{3u}{4} + \frac{1}{2} \, u \left[\log \left(\frac{1 + \sqrt{1-u}}{\sqrt{u}} \right) \right] + \left(\frac{1 - \sqrt{1-u}}{2} \right)^2 .$$

The density is

$$g_3(u) = \tfrac{1}{2} + \tfrac{1}{2} \log\left[\frac{1 + \sqrt{1-u}}{u}\right] \quad .$$

It is easy to check that g is decreasing, $g_3(0) = +\infty$, $g_3(1) = \tfrac{1}{2}$ and $g_3'(0) = g_3'(1) = -\infty$ .. It is tempting to conjecture that this behavior is true in general, i.e., $g_n(0) = +\infty$, $g_n'(0) = g_n'(1) = -\infty$ for $n \geq 3$.

TABLE 1 [†]

Percentage Points
for $-\log \Lambda_n(\underline{Y})$

Sample Size n	Percentiles			
	.01	.05	.90	.99
2	0.015	0.072	3.3	4.85
3	0.2:6	0.480	5.25	7.25
4	0.6	1.1	6.9	9.0
5	1.2	1.8	8.6	10.8
6	1.7	2.5	10.1	12.6
7	2.4	3.3	11.7	14.3
8	3.0	4.2	13.2	15.8
9	3.7	5.0	14.7	17.6
10	4.6	5.8	16.1	19.2

[†] Note that we use lower percentiles for testing exponentiality versus IFRA and upper percentiles for testing IFRA versus DFRA.

TABLE 2

Percentage Points for the IFR
Likelihood Ratio Statistic $\Lambda_n^*(\underline{Y})$

Sample Size n	Percentiles .05	.01	Number of Random Simulations Used
2	.0500	.0100	50,000
3	.025	.004	50,000
4	.0162	.0027	40,000
5	.0125	.0017	50,000
6	.01	.0015	60,000
7	.0087	.001	60,000
8	.0077	.001	80,000
9	.0065	.0007	70,000
10	.0055	.0007	50,000

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6.   Comparisons with Competing Tests

In a paper in process (Barlow and Jacobson) we study in some detail the
robustness of the IFRA and IFR likelihood ratio tests relative to selected competing
tests. Preliminary investigations indicate that the IFR likelihood ratio test is
much better than the Proschan-Pyke test for small samples. However, they have
achieved a remarkable amount of information concerning the asymptotic behavior of
their test statistic. In particular they show that their test statistic, suitably
normalized, has an asymptotic normal distribution for a wide class of alternative
distributions. On the other hand, the distribution of the IFR likelihood ratio
statistic, under the null hypothesis, converges to a weighted sum of χ^2 distributions
which is rather cumbersome to compute. For small sample sizes (say $n \leq 10$) both
statistics are relatively easy to compute with the aid of a desk calculator.

Figure 1 is a graph of the power functions for both tests against Weibull
distribution alternatives (i.e., $F(x) = 1 - e^{-(\lambda x)^{\alpha}}$ for $x \geq 0$ where α is
the shape parameter) when the sample size is 10 and the significance level is
5%. These curves were obtained by means of Monte Carlo simulation on a computer.
The power of the Proschan-Pyke test increases very slowly as a function of the
Weibull shape parameter, α . Even when the shape parameter $\alpha = 3$, the power is
only .62. Our numerical investigaions indicate that for $\alpha > 3$ the power increases
even more slowly and that for $\alpha = 500$, the power is only .86. For $n = 20$,
the power of the Proschan-Pyke test is much better; it yields a power of .92 for
$\alpha = ?$. It should be noted, however, that the power of this test is still not as
good as the power of the likelihood ratio test when $n = 10$.

The asymptotic relative efficiency of the Proschan-Pyke test relative to the
Weibull likelihood ratio test when the true distribution is of the form $G(x) =$
$1 - e^{-(x)^{\alpha}}$ $(\alpha > 1)$ for $x \geq 0$ was computed to be .59 (see Proschan-Pyke (1965)).
The asymptotic relative efficiency of their test relative to the gamma likelihood

ratio test when the true distribution is the gamma is only .20. Unfortunately, asymptotic evaluation of the IFR likelihood ratio test seems to be extremely difficult.

There are many additional unbiased tests of exponentiality versus IFRA or IFR which should perhaps be considered. Pecall that all of the associated statistics are necessarily homogeneous. A statistic related to the IFR likelihood ratio statistic is

$$\Lambda_n^{**} (\underline{X}) = \left(\frac{n}{\frac{n}{\sum\limits_1 X_i}} \right) \prod_{i=1}^{n} (n - i + 1) (X_i - X_{i-1}) \ .$$

If there are no reversals of the normalized differences (they should decrease under IFR alternatives) then Λ_n^{*} and Λ_n^{**} agree except for the factor $n X_1$ and a constant. If $G^{-1} F$ is convex, then

$$\Lambda_n^{**} (\underline{X}) \underset{st}{\leq} \Lambda_n^{**} (\underline{Y}) \ .$$

The test which rejects exponentiality when $\Lambda_n^{**} (\underline{X})$ is sufficiently large is related to a test derived by Moran (1951) for a problem concerning renewal processes. Under the assumption of exponentiality

$$W = \frac{-2 \ \log \Lambda_n^{**}(\underline{Y})}{1 + \frac{n+1}{6n}}$$

is asymptotically distributed as a χ^2 variable with $n - 1$ degrees of freedom. Epstein's (1960) test 8 uses this statistic. Monte Carlo experiments by Zelen (1961) indicate that the power of this test is less than that of the Proschan-Pyke test for small samples against Weibull distribution alternatives.

In section 2 we proved that $F \leq G$ implies

$$\Lambda_n\,(\underline{X}) \underset{st}{\geq} \Lambda_n\,(\underline{Y})\ .$$

Hence we could consider the test, φ^{***}, which rejects exponentiality in favor of the IFRA hypothesis when $\Lambda_n(\underline{X}) \geq c_{1-\alpha}$ where

$$P_G\left\{\Lambda_n(\underline{Y}) \geq c_{1-\alpha}\right\} = 1 - P_G\left\{\Lambda_n(\underline{Y}) \leq c_{1-\alpha}\right\} = \alpha\ .$$

For this test we would use the upper percentile points of $-\log \Lambda_n(\underline{X})$ given in Table 1. Since $\Lambda_n(\underline{X})$ is essentially the maximum likelihood under the IFRA assumption it is perhaps not too surprising that it seems superior to the IFR likelihood ratio test (see Fig. 1). On the basis of computer calculations we conjecture that $-\log \Lambda_n(\underline{Y})$, suitably normalized, is asymptotically $N(0, 1)$.

Perhaps a better test than all of those considered so far is a <u>uniform conditional test</u> [see Cox and Lewis (1966) p. 153] based on the mean of the rectangular distribution. This has been described by Bartholomew as the oldest known statistical test [see discussion in Cox (1955)]. Epstein (1960) adapted this test to the life testing problem and called it test 3. The test is based on the total time on test up to the i-th order statistic $(i = 1, 2, \ldots, n)$, i.e.,

$$T\,(X_i) = \sum_{j=1}^{i}\ (n - j + 1)\ (X_j - X_{j-1})\ .$$

The test statistic is $\sum_{i=1}^{n-1} T(X_i)/ \sum_{1}^{n} X_i$. Under the exponential hypothesis

$$Z = \frac{\sum_{1}^{n-1} T(X_i) - \frac{(n-1)}{2} \sum_{1}^{n} X_i}{\sum_{1}^{n} X_i\ \sqrt{(n - 1)/12}}$$

is approximately $N(0, 1)$ even for relatively small n . If $F \leq G$, then it follows from Theorem 3.12 (iii) [Barlow and Proschan (1966)] that

$$\frac{\sum\limits_{1}^{n-1} T(X_i)}{\sum\limits_{1}^{n} X_i} \geq_{st} \frac{\sum\limits_{1}^{n-1} T(Y_i)}{\sum\limits_{1}^{n} Y_i} \quad .$$

Hence a natural test, φ^{***} , rejects exponentiality in

favor of IFRA if
$$\sum_{1}^{n-1} \frac{T(X_i)}{\sum\limits_{1}^{n} X_i} \geq c_\alpha$$

where
$$P_G \left\{ \sum_{1}^{n-1} T(Y_i) / \sum_{1}^{n} Y_i \geq c_\alpha \right\} = \alpha \quad .$$

Empirical sampling by Zelen and Dannemiller [(1961), p. 47] indicates that this test is superior to φ^{**} against Weibull distribution alternatives. Investigations by Cox (1955) show that the analogue of this test for randomness in a sequence of events is the most powerful test of the Poisson hypothesis against the alternative of a time-dependent Poisson process with occurrence rate

$$\lambda(t) = e^{\alpha+\beta t} \quad .$$

See Bartholomew (1956) for further results concerning this test.

7. Concluding Remarks

It is perhaps worth noting that the percentage points in Table 2 and the results of section 5 also apply to the Boswell test for trend in a stochastic process of Poisson type. However, if the sample size is n and one is using the Boswell statistic then one should locate percentage points in Table 2 corresponding to

the number $n + 1$. A proof for unbiasedness of the Boswell test can be made, patterned after the techniques of section 4.

The number of possible likelihood ratio tests which may be constructed using the definitions in section 1 is fairly large. Recall that the DFR (for decreasing failure rate) maximum likelihood estimate is absolutely continuous when $F(0) = 0$ [Marshall and Proschan (1965)]. Hence one can construct a likelihood ratio test for the following problems:

(1)   versus H_0: F a truncated exponential

                           H_1: F DFR and then IFR $(F(0) = 0)$

(2)   versus H_0: F IFR

                           H_1: F DFR and then IFR $(F(0) = 0)$.

Note that the maximum likelihood estimates under both the hypothesis and the alternative in each case will be absolutely continuous except at the largest observation, X_n , if we impose the additional restriction $F(0) = 0$.

Clearly we can also construct a maximum likelihood test for

(3)   versus H_0: F truncated DFR

                           H_1: F DFR and then IFR.

There is no difficulty in constructing maximum likelihood tests for the problems:

(4)   versus H_0: F exponential

                           H_1: F DFR $(F(0) = 0)$

and

(5)   versus H_0: F DFR $(F(0) = 0)$

                           H_1: F has decreasing density $(F(0) = 0)$.
                                     and F not DFR

The maximum likelihood estimate assuming a decreasing density is given by Grenander (1956). Recall that if F is DFR, then it has a decreasing density.

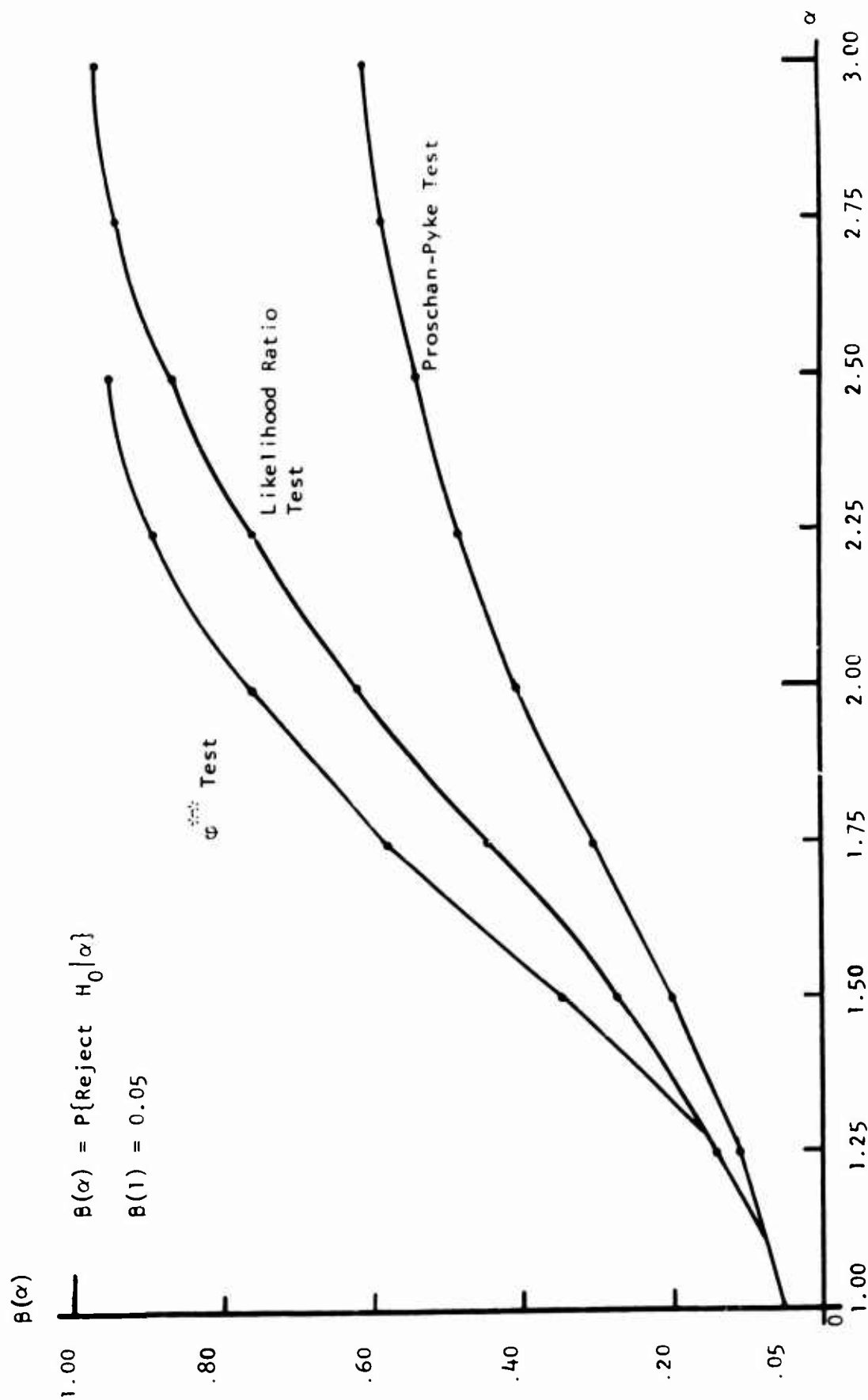Likelihood ratio tests for the two sample problem will be considered in a subsequent paper.

Fig. 1. Power comparison of IFR likelihood ratio test, Proschan-Pyke test and φ^{*} test against Weibull distribution alternatives. (α is the Weibull shape parameter.)

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